Tentamen Computational Methods of Science
November 2, 2012

Duration: 3 hours.

In front of the questions one finds the weights used to determine the final mark.

Problem 1

a. [3] Bring the following equations defined on [0,1] to diagonal form and determine where and which boundary conditions have to be applied (assume that $u(x,0)$ and $v(x,0)$ are given):

$$\begin{aligned} u_t &= u_x - 2v_x \\ v_t &= -2u_x + v_x \end{aligned}$$

b. [2] Write the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \sin(xy)$$

in the form div$A$grad$u = \sin(xy)$, where A is symmetric and determine the type of this equation. Why do we discern elliptic, parabolic and hyperbolic equations?

c. [3] Consider the problem

$$\begin{aligned} \frac{d}{dx}((1+x^2)\frac{d}{dx}u) - u &= f \text{ in } [0,1], \\ \frac{du}{dx} + u &= g \text{ for } x = 1, \\ u &= 3 \text{ for } x = 0. \end{aligned}$$

Give the weak form of this problem to which we can apply the Lax-Milgram theorem and distinguish the bilinear and linear form in this.

Problem 2

a. [3] Give the the finite volume discretization of the equation from part 1.b. in stencil form for the interior of the domain for a grid with equal mesh sizes in both directions.

b. [3] Suppose for the previous part a vertical grid line is coinciding with the left boundary. Give a discretization for a condition $u - u_x = 3$ at this boundary. Next suppose that the left boundary of the domain is precisely in the middle between two neighbouring vertical grid lines. Give a discretization for the same condition for this grid layout.

c. [2] Consider the problem: Find u in linear space V such that for all $v \in V$ it holds that

$$a(v,u) = F(v)$$

where F and a are a linear and bilinear form respectively and V is such that all elements of it satisfy the essential boundary condition (in homogeneous form). Suppose V_h is spanned by the basisfunctions $\phi_1(x), ..., \phi_N(x)$ (all in V). Derive the linear system that needs to be solved to find the Galerkin approximation of the solution.

Exam questions continue on other side

Problem 3

a. [3] Consider the equation $u_t = u_{xx}$ on $x \in [0, 1]$ and $t > 0$, $u(0, t) = u(1, t) = 0$ and $u(x, 0) = \sin(\pi x)$. Derive the system of ordinary differential equations that arises if we use finite differences in space.

b. [2] Determine the Fourier eigenvalue of the difference operator $\mathcal{S}$ implicitly defined by $\mathcal{S}u_j = (u_{j+1} + 2u_j + u_{j-1})/4$. This operator can be used to smooth a grid function, explain why.

c. [3] Locate the eigenvalues of the following matrix as good as possible (without actually computing them)

$$A = \begin{bmatrix} -2 & 0 & 2 & & \\ 0 & -2 & 2 & & \\ 2 & 2 & -4 & 1 & \\ & & & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}$$

Suppose that A is occurring in the equation

$$\frac{d}{dt}u = Au + f.$$

What is the use of locating the eigenvalues for time stepping methods applied to this equation?

Problem 4

a. [3] Consider the problem $Ax = b$ with A nonsingular. Suppose we want to find an approximate solution of this equation with an $x \in \mathcal{V}$. For which A can we write this as a minimization? Give also the form that should be minimized. What is the associated projection form of this minimization? This projection form is also appropriate for a wider class of matrices A, which class?

b. [3] Suppose a square matrix A of order n has only three different eigenvalues and can be diagonalized. Show that any vector in R^n can be expressed as a sum of three eigenvectors corresponding to the three different eigenvalues, respectively. Next show from this, that three steps of Arnoldi are enough to find the three eigenvalues.

c. [2] Consider the generalized eigenvalue problem

$$Ax = \lambda Bx.$$

where A and B are general n by n matrices and B singular. Which transformed matrix can be used in a power method or the Arnoldi method to find the the eigenvalues λ closest to σ?